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THERMAL CONTRACTION OF MERCURY

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ABSTRACT

The thermal conduction problem for Mercury with radioactive material through a horizontal layer is solved in terms of error functions; and the resulting secular decrease of moment of inertia about the axis of rotation due to thermal contraction is estimated. It is shown that the initiation of the tidal evolution on Mercury was completely balanced with the effect of the thermal contraction during the epoch of cooling in a period of about 48 million years.

THERMAL CONTRACTION OF MERCURY

INTRODUCTION

Liu (Liu, 1968) has suggested that the trapping of the rotation of the planet Mercury was originally affected by the thermal expansion or contraction of its figure during solidification. According to this hypothesis, Mercury was originally at a very high temperature, and different parts have since cooled by different amounts. Two bulges in a state of thermal strain were thus set up on the surface. This suggestion, as contained in Liu's paper (Liu, 1969), was based on results of quantitative calculations. In the present paper, the derivation of the results on the underlying physical problem is given. On the basis of the analysis in the course of this study, it is shown that the thermal process on the planet Mercury during the epoch of solidification in cooling is quite adequate to account for the origin of trapping.

COOLING

Apart from certain complications the radiation from the surface of Mercury when in a liquid state could have disposed of the internal heat in a very short time (Liu, 1969). Liquids in general contract and become denser as they cool, and the material cooled by radiation at the surface sank through the hotter liquid below, thus maintaining irregular convection currents and a continual supply of heat to the surface. In this stage, when the surface temperature was high, the loss of heat by radiation was so rapid that radioactivity could not be supposed to have delayed the formation of the crust. But as soon as the crust of Mercury became so stiff as to stop convection radioactivity would become a controlling influence. The equation of heat conduction is (Jeffreys, 1959)

$$\rho c \frac{dT}{dt} = \frac{\partial}{\partial \chi}\left(k \frac{\partial T}{\partial \chi}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + A \tag{1}$$

where T is the temperature, ρ the density, c the specific heat, k the thermal conductivity and A the rate of generation of heat per unit volume. If the axis of x is taken vertically downwards, the derivatives with regard to the coordinates y and z can be neglected. Further, if the heat conductivity is assumed to be uniform, the condition becomes

$$\frac{\partial T}{\partial t} - h^2 \frac{\partial^2 T}{\partial x^2} = \frac{A}{\rho c} \text{ when } 0 < x < \ell \tag{2}$$

and

$$\frac{\partial T}{\partial t} - h^2 \frac{\partial^2 T}{\partial x^2} = 0 \text{ when } x > \ell \tag{3}$$

where

$$h^2 = \frac{k}{\rho c}$$

and ℓ is the depth of the radioactive layer of Mercury.

Since

$$T = -\frac{A(x - \ell)^2}{2k}$$

satisfies Equation (2), one obtains

$$T = u + \frac{A[\ell^2 - (x - \ell)]^2}{2k} \text{ when } x < \ell$$

and

$$T = u + \frac{A\ell^2}{2k} \text{ when } x > \ell$$

The value of u, being specified for all values from $-\infty$ to $+\infty$, is

$$u = \int_{-\infty}^{\infty} dq e^{-q^2} \phi(x + 2qh\sqrt{t})$$

which will necessarily satisfy the following equation (Fourier, 1955)

$$\frac{du}{dt} = h^2 \frac{d^2u}{dx^2}$$

The boundary conditions are such that at t = 0

$$u = \mu x + S - \frac{A[\ell^2 - (x - \ell)^2]}{2k} \text{ when } x < \ell$$

and

$$u = \mu x + S - \frac{A\ell^2}{2k} \text{ when } x > \ell$$

By specifying $\phi(x) = \phi(-x)$, one finds

$$T = \mu x + \left(S - \frac{A\ell^2}{2k}\right) \operatorname{erf}\left(\frac{x}{2h\sqrt{t}}\right) + \frac{A\ell^2}{2k} - q \qquad (4)$$

where

$$q = \frac{A(\ell - x)^2}{2k} \text{ when } x < \ell$$

and $q = 0$ when $x > \ell$.

In Equation (4) the error function is defined by

$$\operatorname{erf} \theta = \frac{2}{\sqrt{\pi}} \int_0^{\theta} e^{-\xi^2} \, d\xi .$$

For A = 7.3 = 10^{-13} cal. cm.$^{-3}$ sec.$^{-1}$, c = 0.2 cal. g.$^{-1}$ deg.$^{-1}$, k = 0.005 cal. cm.$^{-1}$ sec.$^{-1}$ deg.$^{-1}$, ℓ = 2.5 × 10^6 cm., and S = 1400 deg., then

$$T = \mu x + L \operatorname{erf}\left(\frac{x}{2ht^{1/2}}\right) + Q - q \tag{5}$$

and

$$\frac{\partial T}{\partial t} = -\frac{L}{\pi^{1/2}} \cdot \frac{x}{2ht^{3/2}} \exp\left(-\frac{x^2}{4h^2T}\right) \tag{6}$$

where

$$h^2 = 0.005 \text{ cm.}^2\text{sec.}^{-1}$$

$$L = 944 \text{ deg.}$$

$$Q = 456 \text{ deg.}$$

$$\mu = 3 \times 10^{-5} \text{ deg. cm.}^{-1}$$

SECULAR DECREASE IN MOMENT OF INERTIA

Throughout the process of cooling, a certain amount of mechanical adjustment must have occurred in the shape of Mercury. Consider a shell of internal radius r and external radius r + dr. The temperature, which is defined by Equation (5), is a function of t and r. The raise of temperature in a definite time is of course a very small negative value. The density of the shell will change at the same time from the initial value ρ to ρ (1 - 3αT) where α is the coefficient of the linear thermal expansion. Let the inner radius change to r (1 + β). The external radius will become

$$r(1 + \beta) + dr \left[1 + \frac{\partial}{\partial r}(r\beta)\right]$$

Hence the mass of the shell after the change of temperature is

$$4\pi r^2 \rho dr \left[1 + 2\beta + \frac{\partial}{\partial r}(r\beta) - 3\alpha T\right]$$

But the mass is unaltered. Thus the equation of continuity is

$$2\beta + \frac{\partial}{\partial r}(r\beta) - 3\alpha T = 0. \tag{7}$$

Given the cooling, this is the equation to determine the value of β subject to the boundary condition that β is zero at the center of Mercury. Therefore for any shell β is determined by the changes of temperature within that shell. If the shell simply expanded independently of the interior, the radius would increase by $\alpha T r$ instead of by $r\beta$, so that the excess

$$r(\beta - \alpha T) = \delta r$$

is due to stretching. From Equation (7), the stretching is given by

$$\frac{d}{dr}(\delta r^3) = - r^3 \frac{\partial(\alpha T)}{dr}$$

Whence

$$\delta = - \frac{1}{r^3} \int_0^r r^3 \frac{\partial(\alpha T)}{\partial r} dr.$$

The changes must take place in a short time dt, then

$$\frac{\partial \delta}{\partial t} = - \frac{1}{r^3} \int_0^r r^3 \frac{\partial}{\partial r}\left(\alpha \frac{\partial T}{\partial t}\right) dr.$$

Let x be the depth of a point below the surface, then $r = r_0 - \chi$ where r_0 is the radius of Mercury. As the temperature change only extend downwards through a small fraction of the radius of Mercury, $(x/r_0)^2$ and terms of higher order can be neglected. Thus

$$\frac{\partial \delta}{\partial t} = - \alpha \frac{\partial T}{\partial t} + \frac{3}{r_0} \int_x^{\infty} \alpha \frac{\partial T}{\partial t} dr \tag{8}$$

Now, the value α can be expressed as

$$\alpha = \epsilon_1 + \epsilon_2 T$$

where ϵ_1 and ϵ_2 are two constants. From Equations (5), (6), (7) and (8), one obtains the amount of stretching at the surface:

$$\delta = - \frac{6Lh}{\pi r_0} \left\{ \epsilon_1 (\pi t)^{1/2} + \epsilon_2 \left[(Q + 2^{3/2}) (\pi t)^{1/2} + \mu h \pi t \right] \right\} \tag{9}$$

For $\epsilon_1 = 7 \times 10^{-6}$ deg.$^{-1}$, $\epsilon_2 = 2.4 \times 10^{-8}$ deg.$^{-2}$, and $t = 1.5 \times 10^{15}$ sec., Equation (9) gives

$$\delta = - 2.8 \times 10^{-3} .$$

From this the increase of the radius of Mercury by thermal expansion during the period of cooling of the shell is at once found to be about -7 km. The rate of increase of the radius is, then, about -4.7×10^{-10} cm. sec.$^{-1}$. Therefore the average rate of change of the moment of inertia about the axis of rotation during the period of cooling is approximately

$$\frac{dC}{dt} = - 3.1 \times 10^{-10} M_m r_0 \text{ g. cm.}^2 \text{ sec.}^{-1} \tag{10}$$

where M_m is the mass of Mercury.

AXIAL ROTATION

During the epoch of solidification in cooling, Mercury would rotate about its polar axis with a time-dependent inertial tensor. The rotation of Mercury was

$$\frac{d}{dt}\left[C_{(t)} \frac{d(f+\phi)}{dt}\right] = -N \tag{11}$$

where N is the couple due to solar tides, f the true anomaly and ϕ the angle of orientation of Mercury relative to the sun. Equation (11) can be rewritten in the following form,

$$\frac{d^2(f+\phi)}{dt^2} = -\frac{N}{C_{(t)}} - \frac{H}{C_{(t)}} \tag{12}$$

where

$$N = \frac{18}{5} k_2 \pi G \rho \frac{M_s^2}{M_m^2} \cdot \frac{r_0^6(1+e\cos f)^6}{a^6(1-e^2)^6} \cdot C_{(t)} \cdot \mathrm{Sin}\ 2\epsilon$$

$$H = \frac{d(f+\phi)}{dt} \cdot \frac{dC_{(t)}}{dt}$$

and a is the major semiaxis, e the orbital eccentricity, G the gravitational constant, k_2 the Love number, ϵ the phase lag of the tides and M_s the mass of the Sun.

For $e = 0.2$, $G = 6.7 \times 10^{-8}$ dyn. cm^2 g.$^{-2}$, $k_2 = 0.02$, $2\epsilon = 0.005$, $\rho = 5.0$ g. cm.$^{-3}$, $M_s/M_m = 6.0 \times 10^6$, $r_0/a = 4.1 \times 10^{-5}$ and $r_0 = 2.42 \times 10^8$ cm., the average value of $N/C_{(t)}$ over a period of revolution is

$$\frac{N}{C_{(t)}} = 0\ (10^{-23})\ \text{rad. sec,}^{-2}.$$

The calculation also includes the reversal of the phase lag during perihelion passage.

At the 3:2 resonance condition, $d(f+\phi)/dt = (3/2) \cdot n = 1.8 \times 10^{-6}$ rad. $\sec^{-1}$. where n is the orbital mean motion. From Equation (10), one obtains

$$\frac{H}{C_{(t)}} = -0\,(10^{-23})\ \text{rad. sec.}^{-2}.$$

Therefore, it is seen that the initiation of the tidal evolution was completely balanced with the effect of the thermal contraction during the epoch of cooling in a period of about 48 million years.

CONCLUDING REMARKS

In the review of the hypothesis of Darwin for the Earth, it should be noted that in the initial stage the thermal contraction of Mercury may be more effective rotationally than the solar tides because the loss of heat at that stage was quite exceptionally great. Therefore the estimation of $H/C_{(t)}$ may be regarded as a lower limit. Since the above calculations are of the same order of magnitude, it seems not improbable to conclude that the influence of thermal contraction provided the conditions for the rotation of Mercury to be trapped into the 3:2 resonance state.

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